Geometric Mechanics for Sand-swimming

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Sand-swimming, a form of desert burrowing, offers interesting potential as a locomotion mode for robots operating in loose, granular environments. Unfortunately, the computational cost of modeling the relevant physics raises obstacles to a thorough exploration of the system dynamics. Geometric mechanics offers techniques for reducing the complexity of evaluating gaits, thereby offering the potential for exploring a gait design space; unfortunately, these tools have historically been restricted to systems with linear, analytical dynamics. In this paper, we present a framework for combining empirical data from nonlinear models with geometric gait evaluation methods. The resulting tools both reduce the computational costs of describing sand-swimming and reveal fundamental aspects of the motion.

Keywords: geometric mechanics, sand-swimming, gait

Introduction

Sand-swimming, a burrowing mode employed by some desert-dwelling animals, offers interesting possibilities for robotic locomotion in similar environments, especially where concealment or harsh surface conditions are a concern. To make use of this mode of locomotion, however, it is necessary to move beyond simple biological inspiration and identify its key physical principles. In our past work,[1,2] we have developed models of sand-swimming that capture the granular flows and forces acting on the locomoting system. These models are, however, computationally expensive, making it difficult to experiment with different body morphologies or gait patterns. In this paper, we develop computational procedures that allow us to both better understand the fundamental physics of locomotion in granular flow and to expediently evaluate gaits so that we can more readily explore the design space of mechanism and controller.

The tools developed in this paper have a solid foundation in geomet-

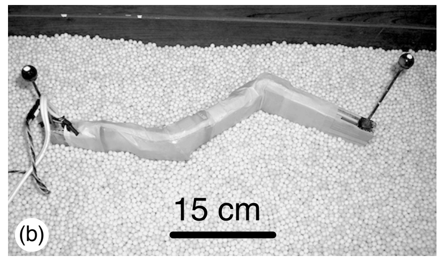

Fig. 1. (a) Sandfish lizard. (b) Three-link robot on bed of plastic particles. LED masts at the end of the robot allow for tracking while robot is submerged.

ric mechanics. This field offers several methods for reducing computational complexity in locomotion problems. First, it introduces notions of *symmetry* that present opportunities for efficient re-use of expensive calculations. Second, these symmetries identify effective gait cycles with geometric features on a system's configuration space,[3] allowing them to be directly identified with minimal trial-and-error iteration. Unfortunately, these methods have historically been restricted to systems with analytically-describable linear dynamics, and have not found wider use in considering systems in non-linear or non-analytic regimes.

To move past these limitations on geometric analysis, and bring their benefits to complex locomotion modes like sand-swimming, we have developed a hybrid approach that uses a geometric framework to organize empirical information gleaned from nonlinear physics models. This approach gives us considerable insight into sand-swimming, highlighting how gait amplitudes and patterns contribute to the displacements they induce. Additionally, the underlying framework is inherently generalizable to other physical domains, suggesting further avenues of exploration.

1. Sand-swimming

Sand-swimming is a form of locomotion exhibited by a variety of desert animals, such as the sandfish lizard in Fig. 1(a). These organisms move below the surface of the terrain, protecting themselves from heat and predators while stalking their own prey. An interesting feature of this motion is that the sand around the animal can act as a granular frictional fluid,[2] making the progress of the organism much more akin to swimming than it is to burrowing or walking. We have investigated sand-swimming at several levels, including computational modeling and both biological and robotic experiments. The key products of this research are a detailed (and compu-

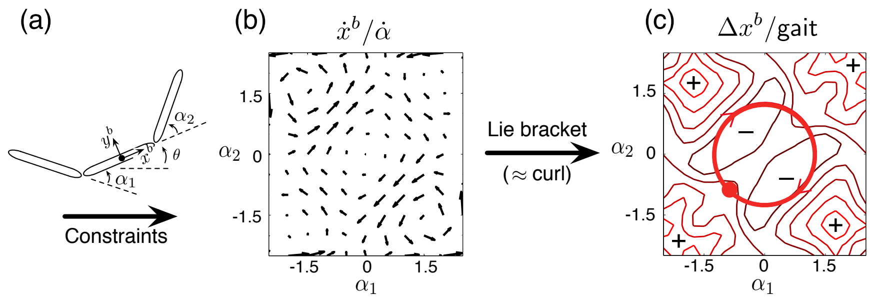

Fig. 2. (a) An example three-link system with joint angles α. (b) Many systems' dynamics can be reduced to linear constraints between a system's body velocity and its shape velocity $\dot{\alpha}$, represented by vector fields. (c) The net displacement from a gait corresponds to the areas it encloses on the components of the *Lie bracket* of the fields (approximately their curls). These plots are for a three-link sandswimmer, and were derived via the method described in §3.

tationally expensive) *discrete element model* (DEM) of the granular flow around a sand-swimmer, capturing the motions of individual grains; a simpler (but still nonlilnear) *resistive force theory* model for the forces on a body moving through a granular medium, fit to empirical data from the DEM; and a robotic test platform, a three-link version of which appears in Fig. 1(b). These tools allow us to evaluate the motion of a sand-swimmer, but the time taken for a given evaluation (on the order of days for a DEM simulation of a given gait) makes extensive trials prohibitively costly.

2. Geometric Mechanics

Geometric mechanics applies principles from differential geometry to problems in classical mechanics.[4,5] A powerful application of these principles to locomotion is the reduction of many mobile systems' equations of motion to a set of shape-dependent linear constraints between its body velocity and its rate of shape change.[3,6–12] These constraints can be visualized as a set of vector fields, like that shown in Fig. 2(b). Applying a function called the *Lie bracket* (conceptually similar to curl) to these fields measures how the constraints vary over different shapes. By Stokes's theorem, the net displacement over any gait corresponds to the area it encloses on these Lie bracket functions.[3] For example, the gait in Fig. 2(c) negatively (counterclockwise) encloses a negative region of the function, and so produces a positive x^b displacement of the system.

Lie bracket approaches are based on an identity between the *exponential* *coordinates** $z(\phi)$ of the net displacement over a gait ϕ (a closed trajectory in the shape space) and a series whose first two terms correspond to the integral of the abstract *curvature* of the constraints over a region of the shape space bounded by ϕ.[13] The Lie bracket measures the *curvature* of the constraints encoded by the local connection, which also corresponds to the net translation induced by a differential oscillation in the system's shape. In two dimensions, the identity appears as

$$z(\phi) = \iint_{\phi} \overbrace{\underbrace{-\mathrm{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{\big[\mathbf{A}_1, \mathbf{A}_2\big]}_{}}^{\text{curvature (Lie bracket)}} \underbrace{d\alpha + \text{higher-order terms},}_{\text{noncommutativity}} \tag{1}$$

where the curl operator is applied individually to each row of $\mathbf{A}$, and $[\mathbf{A}_1, \mathbf{A}_2]$ is the *local* Lie bracket of the columns of $\mathbf{A}$ (taken as if $\mathbf{A}$ did not depend on the shape), which evaluates on $SE(2)$ as

$$\big[\mathbf{A}_1, \mathbf{A}_2\big] = \begin{bmatrix} a_{y1}a_{\theta 2} - a_{y2}a_{\theta 1} \\ a_{x2}a_{\theta 1} - a_{x1}a_{\theta 2} \\ 0 \end{bmatrix}. \tag{2}$$

In (1), the curl term measures the *nonconservativity* of the local connection, or how the constraints change over the shape space, preventing antipodal segments of a gait from pushing or pulling the system equally. The local Lie bracket and higher order terms correspond to the *noncommutativity* of the system's position space, *i.e.*, the extent to which translations with intermediate rotations do not commute, as in parallel parking maneuvers.

3. Hybrid Emprical-Geometric Mechanics

Geometric constraint models like those shown in Fig. 2 both improve the speed of computation—they are typically faster to evaluate than the raw forcing equations on a given system—and allow for simple characterization of gait effects via the Lie bracket. In past work, however, it has been assumed that these constraints can only be generated from analytically-describable linear physics models, such as the conservation-of-momentum

*The exponential coordinates of a position are the components of the constant body velocity required to reach that position in unit time, starting from the origin. A mapping between exponential coordinates and displacements is provided in various sources,[11] but for the purposes of this paper, it is sufficient to note that this mapping is an identity mapping for pure translation or pure rotation; i.e. $\exp([z_x, z_y, 0]^T) = (z_x, z_y, 0)$ and $\exp([0, 0, z_\theta]^T) = (0, 0, z_\theta)$.[11]

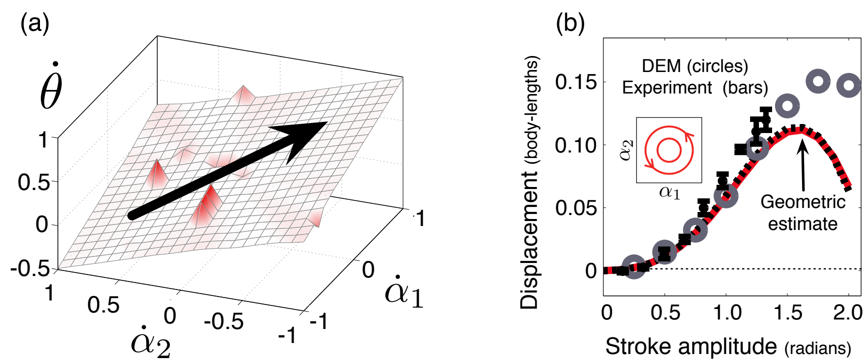

Fig. 3. (a) The body velocity of the sand-swimmer is empirically a linear function of the shape velocity $\dot{\alpha}$. (b) Geometric analysis of an empirically-fit sandswimmer model gives good accuracy, at a fraction of the computational cost.

constraints acting on a satellite or the low-Reynolds number flows around a micro-organism. Granular flow produces nonlinear forces that must often be evaluated through simulation, which would initially seem to disqualify it from such analysis.

In our work with sand-swimmers, however, we have observed that these nonlinear forces combine to produce nearly-linear system velocities, as shown in Fig. 3(a). Fitting linear functions to this velocity relationship (at a reasonably-dense sampling of shapes) produces a geometric constraint model that approximates the behavior of the sand-swimmer. This constraint model is orders of magnitude faster to evaluate than the DEM (seconds verses days), and, as shown in Fig. 3(b), gives an accurate approximation of the system motion for gaits with all but the largest joint motions. Further, by observing patterns in the Lie bracket function, we can see *why* the displacement grows and falls off with amplitude as it does: small amplitude gaits enclose sign-definite areas, so increasing the amplitude increases the net displacement, while very large gaits enclose both positive and negative areas, and so do not experience the same growth in displacement.

4. Optimal Coordinates

A fundamental problem with integrating the motion of the system via Lie brackets is that they do not completely measure the net displacement; instead, they measure the "forwards minus backwards" motion in each body direction, together with a first-order approximation of the non-commutative effects of rotation coupled with translation. Although planar rotations always commute with each other and are unaffected by translations, translations specified in body coordinates only commute when there are no intervening rotations. For example, moving a system forward and backward by one unit each returns it to the origin, but interposing a rotation between the two steps leaves the system with a net translation. This noncommutativity can be captured

This lack of commutativity was long viewed as inherent to the locomoting systems.[11] In our work on nonholonomic systems, however, we observed that in many instances the noncommutativity can be alleviated by an appropriate choice of body frame for the system.[3] In these choices of coordinates, the reference line on the system used to represent its orientation θ rotates very little in response to changes in shape, allowing the translation motions produced by those shape changes to "almost commute."

The optimal choice of body frame for a system are those for which the vector fields encoding the constraint are "smallest" – i.e. those for which a given change in shape produces the least translation and rotation of the system. We can directly solve for these optimal coordinates by first making an arbitrary choice of body frame (the center link is often a good choice, as it allows for convenient representation of the system kinematics, and finding the constraint vector fields for that frame. We can then find the body frame (out of the set of all body frames) whose motion relative to the first link during shape changes most cancels out the motion induced by the constraints. This operation takes the form of finding (conservative) gradient fields that are as close as possible to the constraint vector fields, as illustrated for orientation in Fig. 4. More details on this process are given in our prior work,[3] including an implementation of body-frame optimization based on a *Hodge-Helmholtz decomposition*.[14]

Conclusions

By combining our geometric and granular modeling tools, we have both increased the speed with which sand-swimmer gaits can be evaluated and gained insight into the forms taken by solutions to the gait equations. As we further explore sand-swimming and related motions, such as rubble-crawling, these tools will allow for rapid iteration of swimmer morphology and gait pattern, directing expensive simulations with the full suite of granular tools to the inputs where they will be most informative. We are also very interested in finding other applications of emperical-geometric models to provide insight into complex problems.

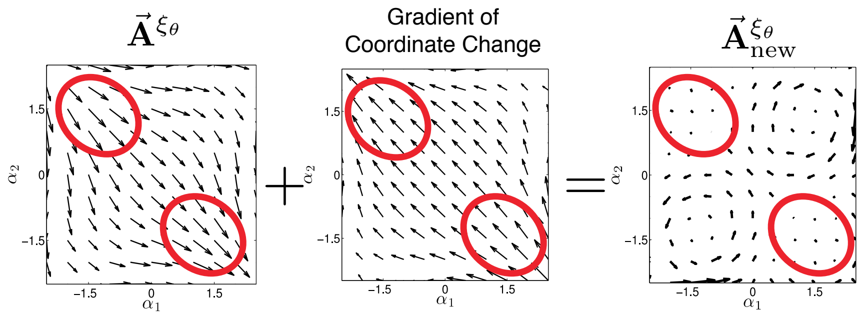

Fig. 4. Calculating the optimal body frame starts with finding the constraints in a convenient body frame, such as the middle link (left). The gradient of the optimal body frame's orientation relative to the middle link is the conservative vector field that is most opposite to the θ constraint field (center). The residual values after summing these two terms (right) are the new minimized constraint field, equivalent to finding the constraints in the optimal coordinates.

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