Absence of inelastic collapse in a realistic three ball model

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Inelastic collapse, the process in which a number of partially inelastic balls dissipate their energy through an infinite number of collisions in a finite amount of time, is studied for three balls on an infinite line and on a ring (i.e., a line segment with periodic boundary conditions). Inelastic collapse has been shown to exist for systems in which collisions occur with a coefficient of restitution r independent of the relative velocities of the colliding particles. In the present study, a more realistic model is assumed for r: $r=1$ for relative velocity equal to zero, and r decreases monotonically for increasing relative velocity. With this model, inelastic collapse does not occur for three balls on a line or a ring. [S1063-651X(98)02404-0]

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I. INTRODUCTION

Energy loss during collisions of macroscopic particles is often described by a coefficient of restitution r, the magnitude of the ratio of the relative normal velocity of the particles after the collision to the relative normal velocity before the collision. Analyses of particle dynamics with constant r have shown that for r below a critical value r_c, many initial particle velocities and configurations lead to an infinite number of collisions in a finite time [1–6]; both the relative spacings and velocities of the balls go to zero. Such a process is called inelastic collapse. In addition to the collapse analyses, simulations [7–9] and hydrodynamic analyses [6,10–17] of granular media have usually assumed r to be constant, *independent* of the relative collision velocity u. For real materials, however, r is not constant; rather, it increases monotonically with decreasing u and approaches unity in the limit that $u \to 0$ [18–21].

To illustrate the problem with the usual assumption of constant r, we examine the two simplest models in which inelastic collapse has been shown to occur for constant r: three balls confined to an infinite line [1,3,4] and three balls confined to a periodic line segment (ring) [5]. For both of these models we find that if r is a physically reasonable function of the relative collision velocity, there is no collapse state. This result builds on a conjecture of McNamara and Young that collapse is an artifact of the idealized constant r model, and that a velocity-dependent r might eliminate this artifact [1].

The reason for the absence of inelastic collapse with a physical model for r is straightforward. If collapse is to occur, the relative velocities of all particles must go to zero. If $r \to 1$ as the relative velocity $u \to 0$, then for u small enough, a collision will occur for which $r > r_c$. From the previous work, this ensures that collapse cannot occur. Thus collapse in the line and ring geometries happens only for nonphysical coefficients of restitution. Therefore the results obtained in recent analyses of inelastic collapse [1–6], as well as work on the hydrodynamics of granular materials [6,10–17], should be reexamined using a more physically accurate form of r.

II. THREE BALLS ON AN INFINITE LINE

Consider three balls of unit mass and labels L, M, and R (left, middle, and right). The balls' velocities are v^L, v^M, v^R, and their relative velocities are $u^L = v^L - v^M$, $u^R = v^M - v^R$. Assume that the balls undergo instantaneous binary collisions and that the relative velocities of two particles before and after their ith collision, u_i and u_{i+1}, are related by a velocity-dependent coefficient of restitution, $r(u_i)$:

$$u_{i+1} = -r(u_i) u_i . \tag{1}$$

Without loss of generality, we assume that the system is prepared such that the velocities of the left and right balls are directed in towards the middle ball, and that the velocities of the balls are such that the left and middle balls undergo the first collision (i.e., $u_0^L > u_0^R > 0$). After the collision between the left and middle ball, the relative velocities are (using the conservation of momentum and the definition of r)

$$u_1^L = -r(u_0^L) u_0^L , \tag{2}$$

$$u_1^R = u_0^R + \frac{1 + r(u_0^L)}{2} u_0^L . \tag{3}$$

The middle and right balls collide next. After the collision, the final relative velocities can be written

$$u_2^L = u_1^L + \frac{1 + r(u_1^R)}{2} u_1^R , \tag{4}$$

$$u_2^R = -r(u_1^R) u_1^R . \tag{5}$$

After this collision, the system will be in a state such that the only possible collision is between balls L and M. If these collide, then the next possible collision will be between R and M. Thus we can generate a map which returns the system to a potential collision between L and M after every two collisions. This is done by substituting Eqs. (2) and (3) into Eqs. (4) and (5) and generalizing to obtain

$$u_{n+2}^L = -r(u_n^L)u_n^L + \frac{1}{2}\left[1 + r\left(u_n^R + \frac{1+r(u_n^L)}{2}u_n^L\right)\right] \times \left(u_n^R + \frac{1+r(u_n^L)}{2}u_n^L\right), \tag{6}$$

$$u_{n+2}^R = -r\left(u_n^R + \frac{1+r(u_n^L)}{2}u_n^L\right)\left(u_n^R + \frac{1+r(u_n^L)}{2}u_n^L\right). \tag{7}$$

The iteration must stop if both $u_n^L < 0$ and $u_n^R < 0$ because then both L and R are moving away from M, and there can be no more collisions (i.e., the range of the map contains points that do not lie within its domain). We now investigate the properties of this map.

The only fixed point of the map is $(u^L, u^R) = (0,0)$, for which the three balls move together with both relative velocities equal to zero. To show this, set $u_{n+2}^L = u_n^L \equiv u^L$ and $u_{n+2}^R = u_n^R \equiv u^R$. Substituting into the above equations, rearranging Eq. (6), and denoting $b = [1 + r(u^L)]/2$ gives

$$3bu^L = u^R + r(u^R + bu^L)(u^R + bu^L), \tag{8}$$

$$u^R = -r(u^R + bu^L)(u^R + bu^L). \tag{9}$$

These yield $bu^L = 0$, so that either $u^L = 0$ or $b = 0$. If $b = 0$, the definition of b implies that $r(u^L) = -1$, which is unphysical. Substituting $u^L = 0$ into Eq. (9) leads to the condition $u^R[1 + r(u^R)] = 0$, giving either $u^R = 0$ or $r(u^R) = -1$. Again, the only physical result is $u^R = 0$.

To explore the long time behavior of the system, we calculate the stability of the fixed point. Writing the map in matrix form for small relative velocities du^L and du^R near the fixed point $(u^L, u^R) = (0,0)$ gives [22]

$$\begin{pmatrix} du^L \\ du^R \end{pmatrix}_{n+2} = \begin{pmatrix} \frac{[1+r(0)]^2}{4} - r(0) & \frac{1+r(0)}{2} \\ -r(0)\frac{1+r(0)}{2} & -r(0) \end{pmatrix} \begin{pmatrix} du^L \\ du^R \end{pmatrix}_n . \tag{10}$$

The eigenvalues of the matrix are

$$\lambda_{\pm} = \frac{1 - 6r(0) + r^2(0) \pm \sqrt{[-1 + 6r(0) - r^2(0)]^2 - 64r^2(0)}}{8}. \tag{11}$$

The linearization of our map recovers the previous result of the existence of a critical r [1,3,4], and shows that for velocity-dependent coefficients of restitution, the only value which determines whether a system will collapse is $r(0)$, the value of r at the fixed point. Substituting $r(0) = 1$ into Eq. (11) gives the complex eigenvalues $\lambda_{\pm} = (-1 \pm i\sqrt{3})/2$. The complex eigenvalues have magnitude unity, which implies neutral stability; hence we must argue further to determine the long time behavior of the linearized map around the fixed point.

Since $r(0) = 1$, the analysis reduces to that for perfectly elastic collisions. If collisions are elastic, a collision between two identical balls acts as if the balls pass through each other. Therefore a maximum of three collisions may occur before the balls move away from the fixed point. As the relative velocities approach zero, the balls act elastically, and the dynamics must result in a state where all relative velocities are negative. Since the linearization of the map is valid for small u, inelastic collapse cannot occur—the balls will never reach a state where all relative velocities and separations are zero. This is because the ranges of both the full and linearized maps contain points that do not lie within their domains.

If $r(0)$ is not unity, but $r_c \leqslant r(0) < 1$, where $r_c \equiv 7 - 4\sqrt{3} \approx 0.0718$, previous analysis has shown that the fixed point is unstable, and collapse cannot occur. Collapse can only occur if $r(0) \leqslant r_c$ [1,3,4]. In experiments, such a situation can never be observed, since for real materials, $r(u) \to 1$ as $u \to 0$.

III. THREE BALLS ON A RING

The result for balls on an infinite line says nothing about what might happen if the balls were not allowed to go to infinity as soon as both relative velocities were negative. Therefore we examine a model which allows continued interaction with neighboring balls, specifically, three balls of equal mass on a ring, i.e., confined to a line segment of unit length with periodic boundary conditions. This geometry does not allow the balls to escape collisions. Note that there is no radial acceleration in this model; the ring merely imposes periodic boundary conditions. Grossman and Mungan [5] have shown that collapse occurs in such a configuration for $r < r_c$.

However, if collapse is to occur on a ring, the distances between the balls and their relative velocities must go to zero, so that one of the particles collides alternately with the other two particles, which do not collide with one another. This situation is indistinguishable from three particles collapsing on an infinite line. Since we have already shown that collapse does not occur on the line, collapse does not occur on the ring.

IV. DISCUSSION

We have shown that inelastic collapse, which was found in previous analyses with a constant restitution coefficient, does not occur with a realistic model for the restitution coefficient. While we have considered only three particle systems, we argue that collapse will not occur in an N particle system. Such systems have been studied [1] for constant coefficient of restitution with N particles on a line, and it was found that when r is near 1, the minimum number of particles necessary to create collapse varies as $-[\ln(1-r)]/(1-r)$. Thus, as $r\to 1$, $N\to\infty$.

Studies predicting inelastic collapse have assumed instantaneous collisions. More realistic models of binary particle collisions would have to account for the duration of collisions (particle contact time), which diverges as $u^{-1/5}$ as $u\to 0$ [19]. Since inelastic collapse requires that the particles undergo an infinite number of collisions in a finite time, collapse cannot occur if the collisions are not instantaneous. For small relative velocities, the duration of the collision significantly affects the particle dynamics. The incorporation of the finite contact time into the analysis complicates the problem because particles are no longer limited to binary collisions. The combined effects of a velocity-dependent coefficient of restitution and finite duration collisions make inelastic collapse in the laboratory unlikely. We note that simulations with a velocity-dependent r [23] and experiments [24] do not produce collapse, but show particle clustering, a situation in which variations in particle density spontaneously occur. It is possible that clustering in granular media proceeds through frustrated collapses, situations in which the collision frequency increases rapidly until the relative normal velocities are such that collapse ceases. However, clustering may also be due to finite duration collisions, or the inelasticity of particles may cause clustering through a scenario less catastrophic than inelastic collapse.

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