Scaling and dynamics of sphere and disk impact into granular media

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Direct measurements of the acceleration of spheres and disks impacting granular media reveal simple power law scalings along with complex dynamics which bear the signatures of both fluid and solid behavior. The penetration depth scales linearly with impact velocity while the collision duration is constant for sufficiently large impact velocities. Both quantities exhibit power law dependence on sphere diameter and density, and gravitational acceleration. The acceleration during impact is characterized by two jumps: a rapid, velocity-dependent increase upon initial contact and a similarly sharp depth-dependent decrease as the impacting object comes to rest. Examination of the measured forces on the sphere in the vicinity of these features leads to an experimentally based granular force model for collision. We discuss our findings in the context of recently proposed phenomenological models that capture qualitative dynamical features of impact but fail both quantitatively and in their inability to capture significant acceleration fluctuations that occur during penetration and which depend on the impacted material.

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I. INTRODUCTION

Collisions with complex particulate materials occur in diverse situations ranging from asteroid impact [1] to the penetration of a running crab’s leg into beach sand [2]. Accordingly, collisions with granular media have long been investigated [3], and, like many areas of granular research, are being actively explored today in experiment [4–11], simulation [12,13], and theory [5–7,10,13–15]. However, because the physics of such events must account for both fluid- and solidlike behavior during impact, understanding remains limited. No comprehensive continuum theory exists for even the relatively low impact velocity of a rock dropped into beach sand from an outstretched hand.

Recent experiments and simulations of low-velocity (≤5 m/s) impact with granular media have mainly considered the depth to which an object penetrates before stopping. These studies have investigated how the penetration depth scales with various system parameters. Durian and co-workers [15,16] performed experiments at low collision velocities $v$ (the maximum penetration depth was approximately a sphere diameter) on a variety of spheres of differing radii $R$, sphere density $\rho_s$, and granular particle density $\rho$ and found that the penetration depth $d$ scaled as $d \sim R^{2/3}(\rho_s/\rho)^{1/2} R^{2/3}$. de Bruyn and Walsh [5] found a different scaling: $d \sim R^{1/2}(\rho_s/\rho)^{1/2}$. The latter experiments were conducted at larger impact velocities and with spheres of higher density such that maximum penetration depths were much greater than the sphere diameter $d \gg 2R$. In two-dimensional (2D) disk simulations, Tsimring and co-workers found that $d \sim v_t^{1.5}(\rho_s/\rho)^{1/2} R^{3/5}$.

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Phenomenological models have been proposed to account for experimental results of penetration depth from spheres impacting granular materials in a gravitational field [5,7,10,14,15]. In all models, the force on a vertically falling object impacting a horizontal granular medium is written in the general form

$$m \frac{d^2 z}{dt^2} = -mg + F_d,$$

where $m$ is the mass of the impactor, $z$ is the displacement of the lowest point on the object below the initial free surface of the grains, $g$ is the gravitational acceleration, and $F_d$ is the drag force due to the presence of the granular medium. Impact models typically represent the drag force as the sum of two terms:

$$F_d = F_z + \alpha v^2,$$

where $v = dz/dt$, $\alpha$ is a drag coefficient, and $F_z$ is posited to be a frictional or hydrostatic term with different proposed forms [14,15] depending on the impact regime and impactor geometry. For example, the authors of [14] assume that an impacting sphere experiences hydrostaticlike forces, basing their arguments on experiments of [17] in which the force on a flat intruder moving at a low constant velocity increases linearly with depth below the free surface. They argue that $F_z$ for shallow impact increases as $F_z \approx 2 \eta p g z^2 R$, with $\eta$ a parameter dependent on grain properties (e.g., angle of repose) and $p$. For deep impact, the free surface is assumed to move along with the sphere such that the bottom of the object is always a fixed distance $z_0$ below the local free surface; thus $F_z$ approaches a constant with value $F_z = 2 \eta p g z_0^2 R$.

Durian and co-workers [15] fitted penetration depth scaling data from experiment to derive a functional form for the frictional drag in the shallow penetration regime such that

$$F_z = mg + mg[3(z/d_0)^2 - 1] \exp(-2|z/d_1|),$$

where $d_0$ and $d_1$ are adjustable fitting parameters.
where \( d_0 \) is the penetration depth for an object impacting the surface with initial collision velocity \( v_c = 0 \), and \( d_1 = m/\alpha \). These two parameters are independent of \( v_c \). Similar to the model of Tsimring and Volkson [14], Eq. (3) also scales like \( z^2 \) for \( z < D \), and approaches a constant \( (mg) \) for sufficiently large impact depth. Recently Durian and co-workers [18] proposed that a linear \( F(z) \) (similar to the model proposed in [19]) provides a better fit.

To summarize, the granular medium is modeled by a force law with a hydrodynamic drag term proportional to the square of the velocity (which dominates at high velocity and thus deep penetration) and a term that accounts for a depth-dependent static resistance force, which dominates at low speeds and thus shallow penetration depths. Other studies have proposed that the drag force also includes a term linear in velocity \([5,6,13]\); however, the experiments in \([6,13]\) are in a much higher-velocity regime, 700 m/s, than the regime examined by us and in the other studies cited here, where impact velocities are typically less than 5 m/s.

Since at least 1742 \([3]\) various force laws for granular impact have been proposed and their associated penetration scalings discussed; however, there have been no detailed experimental three-dimensional studies of the forces that the colliding object experiences during impact. Measuring and understanding the forces exerted during impact is clearly important, as demonstrated by the recent and surprising experimental finding that a disk comes to rest in a time \( t_c \) independent of the initial impact velocity \([4]\). Fits of position versus time indicated that the acceleration during penetration was constant with magnitude dependent only upon \( v_c \).

Accordingly, we describe here direct measurements of the forces exerted on a sphere during penetration of a granular medium. Integrated force measurements show how the penetration depth scales in deep impact experiments (our data most closely follow de Bruyn and Walsh’s scaling \([5]\)) while a systematic study of collision duration reveals simple scaling with system parameters. We examine key features of the dynamics using the acceleration data and then use them to guide an examination of the forces at both high and low depths and velocities. Constrained by experimental force data, we propose an equation to describe the forces during granular impact. Finally, we show that while our impact force equation and those referenced above account for some of the features we observe, a wealth of dynamics associated with force fluctuations remains to be understood.

II. EXPERIMENTAL DETAILS

Figure 1 shows a sketch of the experimental apparatus. A MEMS IC accelerometer (Analog Devices ADXL150) with a range of \( \pm 50 g \)’s was mounted on an aluminum plug, which was inserted into a hole or glued to a flat on top of the impactor; a small-diameter tube isolates the accelerometer wires from the granular medium. We note that the small tube does not affect the impact dynamics because the collapsing crater does not contact the tube until after the impactor has stopped (see top panels of Fig. 1) \([20]\). We choose acceleration to be positive in the direction opposite gravity. Impactors were dropped into the granular medium from heights of \( 0.01 \sim 2.5 \) m with corresponding impact velocities \( v_c \) ranging from 0.4 to 7 m/s.

A variety of impactors and granular materials were used (see Tables I and II). Since holes were drilled in some spheres while flats were made on others and the mass of the accelerometer is included in the indicated sphere mass, the “effective” sphere density \( \rho_{bulk} \) calculated using the masses and radii in the tables does not necessarily match the density of the material of which the sphere is composed. Additionally, six 1.3-cm-radius brass cylinders with different masses were

![Figure 1](image)

FIG. 1. Three sequential images (overhead view) of an \( R = 1.91 \) cm steel sphere impacting glass beads at \( v_c = 2.86 \) m/s with (a) the acceleration and (b) the velocity and penetration depth of the sphere during the collision [the three dots (●) in (a) correspond with the images]. The depth is defined as the distance from the lowest point on the sphere to the initial free surface of the grains. Also in (a) is a sketch of the instrumented projectile (not to scale) showing a single-axis accelerometer embedded in a sphere.

<table>
<thead>
<tr>
<th>Material</th>
<th>Size (mm)</th>
<th>( \rho_{bulk} ) (g/cm(^3))</th>
<th>( \theta_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass spheres</td>
<td>0.25–0.42</td>
<td>1.56</td>
<td>23°</td>
</tr>
<tr>
<td>Aluminum shot</td>
<td>1 x 1</td>
<td>1.62</td>
<td>31°</td>
</tr>
<tr>
<td>Millet seed</td>
<td>1.2</td>
<td>0.72</td>
<td>28°</td>
</tr>
<tr>
<td>Bronze spheres</td>
<td>0.05 and 0.17</td>
<td>5.49</td>
<td>24°</td>
</tr>
</tbody>
</table>
TABLE II. Impactors.

<table>
<thead>
<tr>
<th>Impactor</th>
<th>Radius (cm)</th>
<th>Mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel sphere</td>
<td>9.5</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>130</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>287</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>531</td>
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<tr>
<td></td>
<td>3.5</td>
<td>1437</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>2099</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>3055</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>4079</td>
</tr>
<tr>
<td>Bronze sphere</td>
<td>1.3</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>201</td>
</tr>
<tr>
<td></td>
<td>2.6</td>
<td>518</td>
</tr>
<tr>
<td>Nylon sphere</td>
<td>1.25</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>18</td>
</tr>
<tr>
<td>Bronze disk</td>
<td>1.0</td>
<td>10</td>
</tr>
</tbody>
</table>

(5 mm high)

mounted atop the 1.9-cm-radius nylon sphere, which varied ρs from 1.88 to 9.31 g/cm³. Various containers were also employed (see Table III), including a 29-cm-diameter by 40-cm-high PVC bucket, a 25-cm-diameter by 30-cm-high aluminum pot, a 50-cm-diameter by 75-cm-high cardboard barrel, and a 10-cm-diameter by 10-cm-high glass jar. Two different procedures were used to prepare the granular material: (1) The container was vigorously rocked from side to side with decreasing amplitude until the surface was level; (2) a sieve with outer diameter approximately equal to that of the container was placed in the bottom of the container, the material poured in, and the sieve pulled slowly to the surface. Both procedures produced reproducible dynamics. The majority of experiments were conducted with bronze, steel, or nylon spheres or the bronze disk impacting the 0.25–0.42 mm glass beads prepared via rocking in the 30-cm-diameter PVC container filled to a depth of approximately 25 cm. Additionally, to vary the effective gravitational acceleration during impact, an Atwood machine was used to drop the bucket containing the granular material with accelerations ranging from zero to nearly –g.

III. IMPACT AND DERIVED QUANTITIES

Figure 1 shows the acceleration, velocity, and position of a sphere colliding with a granular medium. We measure the acceleration directly and integrate to obtain position and velocity. At impact, the acceleration increases rapidly as the material suddenly applies force. The acceleration decreases as the sphere penetrates into the medium and finally comes to rest at a finite penetration depth d in a time t½. We discuss the details of the acceleration profile in Sec. IV. We begin with a discussion of the dependence of d and t½ on the initial impact velocity v½ as well as sphere density ρs, sphere radius R, and gravitational acceleration g.

A. Penetration depth

Previous studies [5,14,15] have discussed the dependence of the penetration depth d on the impact velocity or, equivalently, the total change in potential energy of the impactor (refer to the discussion in the Introduction for scalings obtained in this earlier work). Figure 2 shows that, for large enough v½, such that a sphere penetrates more than approximately its radius, d increases linearly with v½. The linear scaling agrees with the data and the scaling proposed in [5]. We do not investigate shallow impact [15] in which this scaling is expected to be modified [14] due to the varying effective cross section of the sphere for z < R.

We systematically vary R and ρs to determine how the penetration depth changes with these parameters. Since our data indicate that the dependence of d on v½ is close to linear for d > R, such that d = d½ + c½ d½, we compute the slope d½ due to the varying effective cross section of the sphere for z < R.

![Penetration depth vs impact velocity for a bronze sphere (R=1.91 cm) impacting glass beads. Solid line (red) is a fit to d=(Δd/Δv½)v½ + d½ with Δd/Δv½=0.0189 s and d½=0.0185 m. Inset: residuals of the fit.](image)

FIG. 2. (Color online) Penetration depth vs impact velocity for a bronze sphere (R=1.91 cm) impacting glass beads. Solid line (red) is a fit to d=(Δd/Δv½)v½ + d½ with Δd/Δv½=0.0189 s and d½=0.0185 m. Inset: residuals of the fit.
as a function of $\rho_s/\rho_g$ and $R$ are obtained with exponents of 0.59 and 0.75, respectively, as opposed to the predicted values (solid curves) of 0.75 and 1. However, comparison of the measured and predicted values of the exponents is not strictly valid, since our fits apply to the $d>R$ region where $d \approx v_c$, whereas those in [15] are for $d<R$ and give the actual penetration depth of a sphere released at the surface (i.e., $v_c=0$).

Combining these scalings, we write the penetration depth for $d>R$ as

$$d \approx \frac{v_c}{R^{3/4}}.$$

FIG. 3. (Color online) (a),(b) Slope and (c),(d) intercept of linear fits of penetration depth vs sphere impact velocity in glass beads for a 1.9 cm nylon sphere (left column) with $\rho_s=1.88, 2.08, 2.85, 3.91, 5.03, 7.97,$ and $9.38$ g/cm$^3$ (blue, red, black, green, cyan, and magenta ○ and blue +) and steel spheres (right column) with $R=0.95, 1.27, 1.51, 1.98, 2.46, 3.49, 3.97, 4.52,$ and $5.00$ cm and corresponding masses $m=34.23, 66.3, 112, 287, 531, 1437, 2099, 3055$ and $4079$ g (blue, red, black, green, cyan, and magenta ○ and blue, red, and black +). The solid curves in (a),(b) are proposed scalings from [5] with $d-d_0 \propto (\rho_s/\rho_g)^{1/2}$ and in (c),(d) from [15] with $d_0 \propto (\rho_s/\rho_g)^{3/4}R$. In (c),(d), the dashed curves are power law fits with exponents 0.59 and 0.75, respectively.

FIG. 4. (Color online) Raw data and collapse using the scalings from Fig. 3. First column (a),(c) Penetration depth and scaled penetration depth vs $v_c$ for varying $\rho_s$. Second column (b),(d) Penetration depth and scaled penetration depth vs $v_c$ for varying $R$. Symbols are the same as in Fig. 3.
where \( C_1 \) and \( C_2 \) are constants. Figure 4 shows both the uncorrected penetration depth data and the collapse obtained using Eq. (4). Although we did not systematically investigate the influence of particle (grain) diameter \( r \) on penetration depth, Eq. (4) suggests \( \Delta d/\Delta v \sim r^2 \) and \( d_0 \sim t_0^{1/4} \).

Figure 5 shows that the penetration depth is affected by finite container size. Decreasing the container diameter by approximately a factor of 2 decreases the slope of \( d \) vs \( v_c \) by about 1.3 for \( v_c \geq 2 \) m/s. There is little apparent difference in \( d \) for smaller \( v_c \). Accordingly, the apparent sublinear dependence of \( d \) on \( v_c \) evident in the collapsed data in Fig. 4(c) might result from the finite diameter of the container, which, as Fig. 5 indicates, becomes more significant the further the impactor penetrates.

\[ d = C_1 v_c \sqrt{\frac{R \rho_s}{g \rho_s^*}} + C_2 (\rho_s/\rho)^{0.59} R^{0.75}, \] (4)

**FIG. 5.** Container diameter effect on penetration depth. A steel sphere with \( R=2.5 \) cm impacting glass beads in cylindrical containers with diameters 28 (●) and 14.6 cm (○). Both containers are filled to a depth of approximately 30 cm. The data (●) are reproduced from Fig. 4. The sketches show the relative sphere and container sizes.

**B. Collision duration \( t_c \)**

1. Lack of dependence of \( t_c \) on \( v_c \)

In a previous quasi-2D study of disks impacting smaller disks confined between narrow sidewalls, Ciamarra et al. found that the collision duration \( t_c \) was independent of \( v_c \) [4]. They attributed this to a constant acceleration during impact whose magnitude was linearly dependent on \( v_c \). Our acceleration data for a 3D disk reveal that \( t_c \) (defined here as the time from impact to arrest; see, e.g., Fig. 1) is nearly independent of \( v_c \) for \( v_c \geq 1.5 \) m/s (see Fig. 6). We denote this velocity-independent collision duration as \( t_0 \). Unlike in [4], but as is the case for spheres, we find that the acceleration is not constant during the penetration phase (see inset of Fig. 6); we return to analysis of the acceleration in Sec. IV. Below \( v_c = 1.5 \) m/s, \( t_c \) decreases and as \( v_c \rightarrow 0 \), \( t_c \rightarrow 0 \). This decrease at low velocity is a consequence of the finite yield stress of granular materials (incorporated in the model of de Bruyn and Walsh [5]), which allows support of a distributed finite load without penetration [21].

We now make an argument for the dependence of \( t_c \) on \( v_c \) for disk impact based on the model of [14] in the deep collision regime. Using Eqs. (1) and (2) with constant \( F_z \), appropriate for the regime of deep impact velocity [14], we write

\[ \frac{dt}{F_z/m + g} + \frac{d}{m v^2}. \] (5)

We supplement this equation with the boundary condition for a disk such that \( v(t=0) = v_c \). Integration yields

\[ t = \frac{\tan^{-1}[v_c/(\alpha(F_c - mg))] + C_1}{\sqrt{\alpha(F_c/m - g/m)}} \]

We define the collision time \( t_c \) as the interval between initial contact and the moment the disk comes to rest (i.e., \( v = 0 \):

\[ t_c = \frac{\tan^{-1}[v_c \sqrt{\alpha/(F_c - mg)}]}{\sqrt{\alpha(F_c/m^2 - g/m)}}. \] (6)

This solution is plotted in Fig. 6 and is a good fit to the experimental data; \( \alpha \) and \( F_z \) are fit parameters. As \( v_c \rightarrow \infty \) for nonzero \( \sqrt{\alpha/(F_c - mg)} \), \( t_c \) approaches \( t_0 = \pi/2 \sqrt{\frac{m}{\alpha(F_c/m-g)}} \), a constant independent of \( v_c \).

The asymptotic form of \( t_c \) given above allows a prediction of the behavior of \( t_0 \) with experimental parameters. Using \( F_z \sim \rho_g R^2 \) and \( \alpha \sim \rho_g R^2 \) as predicted in [6,14], \( t_0 \) should scale as

\[ t_0 \sim \frac{(\rho_s/\rho_g)}{R/g}. \] (7)
This equation is expected to depend on impactor and grain geometry, density, friction coefficients, normal dissipation, and other material parameters.

2. Scaling of $t_0$ in sphere data

The experimental result for the disk data showing that $t_c \rightarrow t_0$ as $v_c \rightarrow 0$, $t_c$ increases with decreasing impact velocity for spheres; compare Figs. 6 and 7 for $v_c \lesssim 1.5$ m/s. We attribute this difference to the fact that, even as $v_c \rightarrow 0$, the small initial contact area between sphere and grains due to the curvature of the sphere always produces local stresses sufficient for grain bed yielding, which consequently allows the sphere to penetrate for a finite distance into the material even with $v_c = 0$; this penetration regime has been examined in [5,16].

For sufficiently high $v_c$, the surface of the sphere in contact with the granular medium is expected to be essentially constant for a large fraction of the collision interval after the initial impact [14]. We therefore expect the proposed scaling of Eq. (7) for disks to be obeyed for spheres as well in the high-$v_c$ regime. As shown in Fig. 8, we find that $t_0 \sim R^{1/2}$ and $t_0 \sim (g_{eff})^{-1/2}$, where $g_{eff}$ is the acceleration of the falling bucket in the Atwood machine. For impact at varying sphere densities, the figure shows that $t_0 \sim (\rho_s/\rho_g)^{1/4}$.

Combining these three scalings, we obtain

$$t_0 \sim \left(\frac{\rho_s}{\rho_g}\right)^{1/4} \sqrt{R/g},$$

or in terms of the sphere mass $t_0 \sim [M/(R^2 \rho_g)]^{1/4}$. A test of this final scaling is shown in Fig. 9 which shows that the $t_c$ data collapse well for a large range of densities and radii. Comparison of the experimentally determined $t_0$ scaling with our model shows that Eqs. (8) and (7) both include a $\sqrt{R/g}$ term but have different exponents in the density term (1/4 and 1, respectively). If these equations are equivalent, the latter result implies that $C$ in Eq. (7) should vary as $(\rho_s/\rho_g)^{-3/4}$.

IV. DYNAMICAL FEATURES OF IMPACT

While $t_c$ and $d$ are characteristic physical quantities associated with any impact into deformable material, we hypothesize that the scaling that matches experimental data can be obtained with a variety of phenomenological models.
In contrast, examine Fig. 10, which shows the acceleration profiles for impact of an \( R = 1.91 \) cm diameter nylon sphere for five distinct impact velocities. As this figure makes clear, while current models capture the average physics involved in the impact events, they miss much of the detailed physics involved in a collision. As an example, we fit the data in Fig. 10 to the model of Ambroso et al. \[15\]. As \( v_c \) increases, the model fails to capture significant acceleration fluctuations as well as the underlying form of the acceleration. Figure 11 shows the corresponding growth in the relative error between experimental data and the model fit as \( v_c \) increases.

The force model from [15] has a second shortcoming: It predicts that the scaling lengths \( d_0 \) and \( d_1 \) in Eq. \( 3 \) are independent of \( v_c \). Least-squares fits of \( a(t) \) for fixed sphere (see references in Sec. I).
mass $M$ while $d_0$ and $d_1$ are allowed to vary are shown for different $v_c$ in Fig. 10. The dependence of $d_0$ and $d_1$ on $v_c$, as for a range of granular materials: A peak in the acceleration during collision, $a_{\text{peak}}$, and a rapid decrease in the acceleration as the object comes to rest, $a_{\text{stop}}$; see Fig. 10. For impact at fixed parameters (i.e., $v_c, R, \rho_s$, and $\rho_g$) Fig. 13 shows that the acceleration profile and associated features are reproducible to within approximately 5% from run to run. We demonstrate how these dynamical features scale with system parameters. Using insights gained from the scaling, we deduce an empirical force model by examining the experimental data in the extremes of high velocity with shallow penetration and low velocity with deep penetration.

A. Peak accelerations

During the collision the acceleration rises to a maximum whose magnitude we denote $a_{\text{peak}}$. At low velocity, the peak is not pronounced, but rather is seen as a broad maximum. As $v_c$ increases, the peak occurs soon after the initial contact of the sphere with the grain surface. For $v_c \geq 1.5$ m/s the interval between contact and peak acceleration is only a few milliseconds. Figure 14 shows that in this regime, for all sphere densities and radii, $a_{\text{peak}}$ increases approximately as $v_c^2$, $a_{\text{peak}} \sim v_c^2$ is in accord with all models $[6,14,15]$ in the high-velocity and low-depth limits, and implies that for these $v_c$ the impact fluidizes grains sufficiently for the system to display inertial fluidlike drag $[22]$. For fixed $v_c$, $a_{\text{peak}}$ can increase by more than 50% if the container is sharply tapped a few times before impact. But, as $v_c \to 0$, the $v_c^2$ scaling does not hold.

FIG. 12. Scaling lengths $d_0$ (●) and $d_1$ (○) in the model of $[15]$ [see Eq. (3)] as determined by best fits to the data set from Fig. 10 are not constants as predicted by the model but depend on the impact velocity.

FIG. 13. (Color online) Details of the acceleration profiles show only small variation (net <5%) from run to run for six separate impacts of an $R = 1.91$ cm bronze sphere into glass with initial velocity $3.65 < v_c < 3.67$ m/s.

FIG. 14. (Color online) Peak impact acceleration $a_{\text{peak}}$ as a function of impact velocity into glass beads for (a) a nylon sphere with effective density 1.88 and 9.38 g/cm$^3$ and $R = 1.9$ cm, and (b) steel spheres with radii $R = 1.3$ and 3.97 cm. Solid lines are fits of $a_{\text{peak}} \sim v_c^2$ for $v_c \geq 1.5$ m/s. The inset shows that, as $v_c \to 0$, the $v_c^2$ scaling does not hold.
of Fig. 14. Figure 10 shows that in the low-velocity limit the peak acceleration occurs at the end of the collision just before the sphere comes to rest. We interpret this as a change from hydrodynamicle like dynamics at high velocities to a regime in which resistance to inertia is no longer the dominant source of drag. This is consistent with Fig. 14(a) which shows the lower-density sphere scaling as $v_c^2$ over a larger velocity range than the higher-density sphere; the latter’s velocity decreases more slowly, resulting in deeper penetration where the influence of $F_z$ is no longer negligible. Since high- and low-velocity regimes are dominated by different physics, we postpone further discussion of $a_{\text{peak}}$ until the treatment of force laws in Sec. IV C.

B. Stopping acceleration

For all $v_c$, the sphere does not come to rest gradually, but instead suffers an abrupt decrease in acceleration before halting, which is reminiscent of a horizontally sliding object stopping due to friction. We denote the magnitude of the decrease as $a_{\text{stop}}$. This impact feature occurs for all spheres, granular materials, and containers we employed in our study. As can be seen in the inset of Fig. 10, $a_{\text{stop}}$ increases with increasing $v_c$. The models of both Tsimring and Volfson [14] and Ambroso et al. [15] predict jumps in the acceleration as the impactor comes to rest, which are attributed to the dominance of the depth-dependent frictional or hydrostatic drag term $F_z$. While the models predict that $a(t)$ instantaneously jumps to zero at the end of penetration, we observe that the rate at which $a(t)$ decreases to zero depends on $v_c$; as $v_c$ increases, the transition to $v=0$ sharpens (see inset of Fig. 10). We comment on this in Sec. IV C.

In Fig. 15 we examine $a_{\text{stop}}$ as a function of penetration depth $d$ for varying sphere density $\rho_s$ and radius $R$, where $d$ is the depth to which the lowest point of the sphere penetrates the material (see Sec. III A). For varying $\rho_s$ [Fig. 15(a)] and constant $R$, we find that $a_{\text{stop}}$ is independent of the sphere density—at the same depth spheres of different density experience the same jump in acceleration (i.e., force is proportional to mass), further evidence for frictional forces dependent primarily on geometry dominating the final stages of penetration. $a_{\text{stop}}$ increases as the ultimate penetration depth of the sphere increases. For $d \geq 2$ cm, $a_{\text{stop}}$ increases approximately linearly, although there is increased scatter toward lower $a_{\text{stop}}$ for $d \geq 7$ cm. For $d \leq 2$ cm, $a_{\text{stop}}$ increases more rapidly with increasing $d$. We attribute this larger slope at small $d$ to the initial growth in contact area between sphere and grains that occurs as the sphere submerges.

For spheres with varying radii and constant density, $a_{\text{stop}}$ also increases with depth, again with increased scatter for larger $d$. Unlike its density independence, $a_{\text{stop}}$ is inversely proportional to sphere diameter at any given depth. Figure 15(b) shows that, when $R a_{\text{stop}}$ is plotted vs $d$, data sets for spheres of various radii fall onto the same curve. This col-
lapse holds for our entire steel sphere data set in which \( m \) varies by more than two orders of magnitude (for clarity, the full data set is not shown). When multiplied by \( R \) the \( a_{\text{stop}} \) data for varying density also fall onto the same curve.

In the final stage of the collision in the vicinity of \( a_{\text{stop}} \), the sphere responds as if it were in contact with an elasticlike medium (see Fig. 16). The sphere does not instantaneously come to rest at \( v=0 \) as predicted by models, but instead the acceleration decreases rapidly (with finite slope) and overshoots (negative acceleration). When the rapid decrease in \( a(t) \) associated with \( a_{\text{stop}} \) begins, the velocity increases through zero, indicating that the sphere has reached its deepest penetration, reversed direction, and is moving upward; for some impact parameters [see Figs. 17(d) and 17(e)], the velocity and position oscillate for a few cycles after the initial overshoot. Recently Durian and co-workers [18] also observed an oscillation in the velocity of a sphere at the end of the collision. They attributed this to displacement of the bottom of the container, but for our data, as seen in Fig. 17(a), the primary overshoot is largely independent of container size and composition for three different containers: cardboard (blue), PVC (red), and aluminum (green) (see Table III for container details). Nor does the interstitial air diminish the effect, as Fig. 17(b) shows for glass beads. Other evidence pointing to an intrinsic origin is provided by Fig. 17(c), which shows that as the sphere density increases at fixed \( R \) the duration of the overshoot decreases, and by Fig. 17(d), which shows that with the same impactor and container the overshoot varies with the granular material. Additionally, the overshoot is observed to decrease with increasing sphere radius [see Fig. 17(e)] but to be largely independent of bed depth for depths ranging from 8 to 20 cm. We speculate that, when the acceleration reaches \( a_{\text{stop}} \), the material suddenly undergoes a solidification transition and the subsequent dynamics are a result of the sphere oscillating within the now elastic solid as Fig. 16(c) suggests and as modeled by [5]. This picture does not explain why the characteristic overshoot time decreases with increasing sphere density.

C. Discussion of force laws

The preceding results for \( a_{\text{peak}} \) and \( a_{\text{stop}} \) suggest that hydrodynamic forces scaling like \( v^2 \) dominate in the high-velocity and shallow-penetration regime, while frictional and...
hydrostatic forces are of primary importance as \( v \to 0 \). To examine these ideas, Figs. 18 and 19 show the velocity dependence of sphere dynamics at various fixed depths during the collision. For shallow depths and varying densities, the accelerations are different but the forces exerted by the grains, \( F=ma-g \), collapse onto a master curve such that \( F \sim v^2 \) [see Figs. 18(a) and 18(d)]. Thus, as assumed by [6,7,10,13–15], a drag force proportional to the velocity squared, and independent of mass, is a good approximation for shallow depths. Additionally, [6,7,10,13–15] assume the force is inversely related to the sphere cross-sectional area. To check this scaling we plot in Figs. 19(a) and 19(c) \( a/g \) vs \( v/\sqrt{Rg} \) and \( F/R^2 \) vs \( v \) at shallow depth and for varying radii. Both quantities fall onto master curves varying as \( v^2 \), indicating that indeed \( F \propto R^2 \).

For deeper penetration and decreasing velocity, the force and acceleration no longer vary as \( v^2 \), but are instead linear in velocity with a nonzero offset at \( v=0 \); see Figs. 18(c), 18(f), 19(b), and 19(d). The slope of \( F \) vs \( v \) is independent of mass [see Fig. 18(f)], suggesting a low-Reynolds-number fluidlike drag. However, as indicated by Figs. 19(b) and 19(d), the linear velocity coefficient of the force varies as \( R^{5/2} \) rather than \( R \) as is the case for Newtonian fluids. Linear velocity dependence has been proposed in [5,6,13], although \( v_c \) was much larger (\( v_c \sim 700 \text{ m/s} \) in [6,13]) than in our study. For both varying density and varying radii, the extrapolated \( v=0 \) intercept occurs at constant acceleration, indicating a force dominated by friction. We note that at very low velocities (\( v \leq 0.25 \text{ m/s} \) \( a \) becomes constant—this is the regime of \( a_{\text{stop}} \) [see Fig. 20(a) for example] and explains why \( a_{\text{stop}} \approx \mu g \).

Summarizing the results of Figs. 18 and 19, we write the drag force \( F_d \) exerted by the granular medium on the sphere as the empirical force law

\[
F_d = \mu(z)mg + C(z)R^{5/2}\rho_g\sqrt{g}v + \alpha'R^2v^2,
\]

where \( \mu(z) \) is a depth-dependent constant analogous to a friction, \( C(z) \) is a drag coefficient also dependent on depth, and \( \alpha' \) is a constant independent of depth. We consider this
the subsequent upward motion of the sphere we mention in our discussion of Eq. 9 is incomplete: the force in the intermediate stages of collision exhibits a more complicated and nonmonotonic dependence on \( v_i \) as seen in Fig. 21. This nonmonotonic behavior may be the result of local changes in the grain layer caused by wave propagation in the bulk. Nor have we examined how \( \mu, C, \) and \( \alpha \) vary with the grain-grain and grain-intruder friction coefficients. For example, preliminary experiments show that, for collision with glass beads, \( t_c \) is reduced by approximately 5% when a monolayer of glass beads is glued to the nylon sphere. Modifications to Eq. 9 will also be required to describe collisions of constant-cross-section intruders (e.g., see the disk data in Sec. III B) and other nonspherical objects.

D. Fluctuations and material dependence

For sufficiently low impact velocity, the model of [15] captures the shape of the acceleration profile (see Figs. 10 and 11). As \( v_i \) increases, the relative difference between the experimental data and the model prediction increases. For low velocity, the acceleration is concave down. According to the model, the force on the sphere increases as the sphere penetrates into the medium, and then smoothly levels out to a constant. As \( v_c \) increases, the curves develop upward concavity. Substantial fluctuations in our experiments appear in the vicinity of the change in curvature, which implies that they are associated with penetration dynamics dominated by the inertial \( v^2 \) term. The models discussed in this paper cannot capture such physics as they are purely hydrodynamic. In addition, Fig. 22 demonstrates that the fluctuations depend on the type of material that the sphere impacts. The fluctua-

![Graph showing acceleration of an R=1.9 cm bronze sphere vs velocity at intermediate stages of collision with glass beads (fixed depth 1.85 < z/R < 1.91) shows nonmonotonic behavior (compare to Figs. 14 and 15).](image)

**Fig. 21.** Acceleration of an R=1.9 cm bronze sphere vs velocity at intermediate stages of collision with glass beads (fixed depth 1.85 < z/R < 1.91) shows nonmonotonic behavior (compare to Figs. 14 and 15).

![Fluctuations in acceleration during penetration depend on the characteristics of the material being penetrated.](image)

**Fig. 22.** (Color online) Fluctuations in acceleration during penetration depend on the characteristics of the material being penetrated. An R=1.9 cm, m=200 g bronze sphere impacting (a) glass spheres, (b) birdseed, and (c) cut aluminum wire (see Table I for material properties). Impact velocities (blue, red, green, black, and magenta) are \( v_i = (a) \ 0.45, 0.84, 1.82, 2.47, \) and 3.40, (b) 0.5, 1.22, 1.74, 2.51, and 3.53, and (c) 0.50, 0.98, 1.79, 2.43, and 3.36 m/s.
tions are significantly more irregular and occur over shorter time scales in millet seeds (smallest density) and aluminum (largest density) than in glass beads, which exhibit a characteristic structure for higher \( v_c \) (see also Fig. 21). We attribute the fluctuations in acceleration to creation and annihilation of elements of the force network [23]. These fluctuations are apparently strongly influenced by particle shape and also size relative to the impactor. They have mostly been observed in the quasistatic regime [24] although quantitative study using photoelastic particles is possible [25].

As Fig. 22 also shows, \( a_{\text{peak}} \) increases with the particle density \( \rho_p \) such that the densest material (aluminum) has a peak acceleration approximately six times that of the millet seeds (birdseed) at a given \( v_c \); the density ratio of the particles is approximately a factor of 2 (see Table 1). However, the magnitude of \( a_{\text{stop}} \) and the overshoot at the end of the collision are largest for the millet seed and smallest for aluminum, which is opposite to the behavior of \( a_{\text{peak}} \). The collision time \( t_c \) decreases monotonically with increasing particle density.

V. CONCLUSIONS

We have directly measured the forces experienced by spheres and disks impacting granular media at collision velocities \( v_c < 5 \text{ m/s} \). By integrating the measured acceleration we have deduced scaling relations for penetration depth and collision time as a function of \( v_c \) and collision impactor parameters and compared them to scalings proposed in the literature. We have identified robust features of the collision dynamics (\( a_{\text{peak}} \) and \( a_{\text{stop}} \)) and described how they scale with \( v_c \) and intruder parameters. These features are in accord with models of drag that propose both inertial and frictional drag terms. We have shown how the combination of such terms determines the surprising finding that the time of collision is independent of velocity. Developing the force law for penetration empirically from our data, we have proposed a model of the drag force on a sphere during impact which includes constant, linear, and quadratic terms, and have shown how the coefficients in this drag relation scale with \( v_c \) and intruder parameters. We have discovered that during impact significant fluctuations in acceleration occur which are not described by any existing models of impact. Whether these material-dependent fluctuations fall outside the scope of continuum dynamics remains to be determined.

The solid volume fraction \( \phi \) was not varied in our experiments. However, previous work on penetration of granular media at constant low velocity [26] revealed a \( \phi \)-dependent transition in penetration force apparently associated with the onset of glassy behavior [27]. Future studies should investigate whether impact dynamics exhibit a similar \( \phi \) dependence.

Our experiments provide another example of the rich dynamics and fundamental physics found in the interaction of objects with granular media in the mixed fluid-solid regime. While there has been much progress in the theory of free granular flow (hydrodynamic regime) [28–31], models describing the mixed regime examined here are not rooted in fundamental physics but are largely phenomenological. Our data can constrain and inform development of models in this regime. We hypothesize that, to accurately capture the dynamics we have observed (including fluctuations and stopping acceleration), the physics of rapid fluidization and solidification must be included. Beyond discrete simulations [4], multiphase modeling like that proposed in [32] might be applicable, or perhaps such features will have to be captured by statistical models.

In addition to probing challenging and unsolved problems in the physics of granular media, models describing the impact regime we have studied here are important to many areas of science and engineering. One area with increasing relevance is control of locomotion in organisms and robots [33,34]. Movement often occurs on complex media [35] and there is a need to understand limb interaction with complex substrates [36]. While the scalings of such properties as penetration depth and collision time might be largely insensitive to intruder and grain geometry, the details of the force developed during penetration certainly are, and thus theory is needed to model such effects.

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